EQUATIONS OF LONG WAVES WITH A FREE SURFACE IV. THE CASE OF CONSTANT SHEAR

Boris A. Kupershmidt †

† *The University of Tennessee Space Institute*
Tullahoma, TN 37388, USA
E-mail: bkupersh@utsi.edu

Abstract

A large class of two-dimensional free-surface hydrodynamical systems is determined that can be self-consistently reduced by the condition that the velocity profile has a constant shear. The reduced systems turn out to be Hamiltonian, and so does the reduction process itself. All reducible systems, Hamiltonian or not, are determined and shown to form a Lie algebra. All this is then generalized to the multilayer/multi-species representations.

1 Introduction

The classical one-dimensional long-wave system

$$h_t = (hu)_x, \tag{1.1a}$$

$$u_t = uu_x + gh_x, \tag{1.1b}$$

has a venerable history. Here $h = h(x,t)$ is the height of a free surface over the bottom $\{y = 0\}$; $u = u(x,t)$ is the horizontal component of velocity; t is the time coordinate (opposite in sign to the physical time); $-\infty < x < \infty$; subscripts t and x denote partial derivatives; g is the gravitational acceleration.

The system (1.1) is Hamiltonian and integrable: it can be put into the form [5,6]

$$\begin{pmatrix} h \\ u \end{pmatrix}_t = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta h \\ \delta H/\delta u \end{pmatrix}, \tag{1.2}$$

where

$$\partial = \partial/\partial x, \tag{1.3}$$

$$H = \frac{1}{2}(hu^2 + gh^2), \tag{1.4}$$

and there exists an infinite number of conserved densities for that system.

In 1973 Benney [1] derived the following *two-dimensional* generalization of the system (1.1):

$$h_t = (\int_0^h u dy)_x, \tag{1.5a}$$

$$u_t = uu_x + gh_x - u_y \int_0^y u_x dy, \tag{1.5b}$$

where now $u = u(x, y, t)$ depends also upon the second space coordinate, $y : 0 \leq y \leq h$. Benney had found two remarkable properties of the *two-dimensional* system (1.5). First, if one introduces the *moments* of the velocity $u(x, y, t)$:

$$A_n = A_n(x, t) = \int_0^n u^n(x, y, t) dy, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.6}$$

then the system (1.5) implies the *autonomous* evolution system

$$A_{n,t} = A_{n+1,x} + gnA_{n-1}A_{0,x}, \quad n \in \mathbf{Z}_{\geq 0}. \tag{1.7}$$

Second, the moments system (1.7) has an *infinite* number of polynomial conserved densities $H_n \in A_n + \mathbf{Q}[g; A_0, ..., A_{n-2}]$:

$$H_0 = A_0, \;\; H_1 = A_1, \quad H_2 = A_2 + gA_0^2, ... \tag{1.8}$$

Subsequently, Manin and myself showed [6,7] that:

(A) The moment system (1.7) is itself Hamiltonian: it can be written in the form

$$A_{n,t} = \sum_{m \geq 0} B_{nm}(H_m), \quad H_m = \frac{\delta H}{\delta A_m}, \tag{1.9}$$

$$B_{nm} = nA_{n+m-1}\partial + \partial m A_{n+m-1}, \tag{1.10}$$

with $H = \frac{1}{2}H_2 = \frac{1}{2}(A_2 + gA_0^2)$, and with the matrix (1.10) being Hamiltonian;

(B) The general system (1.9) is implied by the following two-dimensional free-surface system:

$$h_t = \partial(mA_{m-1}H_m) \tag{1.11a}$$

$$u_t = \partial(u^m H_m) - u_y \int_0^y dy (mu^{m-1}H_m)_x. \tag{1.11b}$$

We sum on repeated non-fixed indices unless directed otherwise;

(C) The Hamiltonians H_k's (1.8) found by Benney are in involution with respect to the Hamiltonian structure (1.10). Therefore, the corresponding higher flows commute;

(D) When u is *y-independent*,

$$u_y = 0, \tag{1.12}$$

so that we are back to the classical one-dimensional case, the map (1.6) becomes

$$A_n = hu^n, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.13}$$

and this map is *Hamiltonian* between the Hamiltonian structures (1.2) and (1.10).

The purpose of this note is to show that there exists an interesting reduced family of the full two-dimensional system (1.11) generalizing the purely one-dimensional reduction $u_y = 0$ (1.12), namely

$$u_y = s = \ const, \tag{1.14}$$

$$u(x, y, t) = v(x, t) + sy, \qquad s = \ const. \tag{1.15}$$

Thus, we consider the case when the shear *is* present but is constant.

We shall verify that: the constraint $\{u_y = s\}$ (1.14) is compatible with the flow (1.11) for *any* Hamiltonian H; that on this constrained submanifold $\{u_y = s\}$ (1.15), the system (1.11) turns into a Hamiltonian system of the form

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & -s\partial \end{pmatrix} \begin{pmatrix} \delta H/\delta h \\ \delta H/\delta v \end{pmatrix}; \tag{1.16}$$

and that the corresponding reduction map

$$A_n = \int_0^h (v + sy)^n dy, \qquad n \in \mathbf{Z}_{\geq 0}, \tag{1.17}$$

is Hamiltonian between the Hamiltonian structures (1.16) and (1.10).

We then determine when similar constant-shear reductions exist for other free-surface hydrodynamical systems.

At the moment, let us record that the original two-dimensional Benney system (1.5) reduces on the submanifold $\{u_y = s,\ u = v + sy\}$, to the system

$$\begin{pmatrix} h_t \\ \\ v_t \end{pmatrix} = \begin{pmatrix} (vh + s\frac{h^2}{2})_x \\ \\ vv_x \ + \ gh_x \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \\ \partial & -s\partial \end{pmatrix} \begin{pmatrix} \dfrac{(v+sh)^2}{2} + gh \\ \\ vh + s\dfrac{h^2}{2} \end{pmatrix}, \tag{1.18}$$

$$H = \frac{1}{2}(A_2^* + gA_0^{*2}) = \frac{1}{2}\Big(\frac{(v+sh)^3 - v^3}{3s} + gh^2 \Big), \tag{1.19a}$$

$$\frac{\delta H}{\delta h} = \frac{(v+sh)^2}{2} + gh, \quad \frac{\delta H}{\delta v} = vh + s\frac{h^2}{2}. \tag{1.19b}$$

2 Constant-Shear Flows

Denote by $(\cdot)^*$ the reduction of the object $(\cdot)$ on the submanifold

$$u_y = s,\ u = v + sy. \tag{2.1}$$

Thus,

$$A_m^* = \int_0^h (v+sy)^m dy = \frac{(v+sh)^{m+1} - v^{m+1}}{s(m+1)}, \quad s \neq 0, \tag{2.2}$$

$$A_{m,v}^* = \frac{\partial A_m^*}{\partial v} = mA_{m-1}^* = \frac{(v+sh)^m - v^m}{s}, \tag{2.3}$$

$$A_{m,h}^* = \frac{\partial A_m^*}{\partial h} = (v+sh)^m, \tag{2.4}$$

$$H_h^* = \frac{\delta(H^*)}{\delta h} = \left(\frac{\delta H}{\delta A_m}\right)^* \frac{\partial A_m^*}{\partial h} = (v+sh)^m H_m^* \tag{2.5}$$

$$H_v^* = H_m^* \frac{\partial A_m^*}{\partial v} = (mA_{m-1}H_m)^* = \tag{2.6a}$$

$$= \frac{(v+sh)^m - v^m}{s} H_m^*. \tag{2.6b}$$

Now, denote temporarily

$$F = u^m H_m, \tag{2.7}$$

so that

$$F_u = mu^{m-1}H_m. \tag{2.8}$$

Differentiating equation (1.11b) with respect to y, we find:

$$u_{y,t} = \partial(F_u u_y) - u_{yy} \int_0^y dy (F_u)_x - u_y (F_u)_x =$$

$$= F_u u_{yx} - u_{yy} \int_0^y dy (F_u)_x. \tag{2.9}$$

When $u_y = s = const, u_{yt} = u_{yx} = u_{yy} = 0$. Thus, the flow (1.11) properly restricts on the constraint $\{u_y = s,\ u = v + sy\}$. Evaluating equation (1.11b) at $y = 0$, we obtain:

$$v_t = \partial(v^m H_m^*). \tag{2.10}$$

Equation (1.11a) becomes:

$$h_t = \partial(mA_{m-1}H_m)^* \ \text{[by (2.6a)]} \ = \partial\left(\frac{\delta H^*}{\delta v}\right), \tag{2.11}$$

which proves the first half of formula (1.16). To prove the second half of that formula, we need to check, in view of the relation (2.10), that

$$\partial(v^m H^*_m) \stackrel{?}{=} \partial(H^*_h) - s\partial(H^*_v) \tag{2.12}$$

or

$$v^m H^*_m \stackrel{?}{=} H^*_h - sH^*_v. \tag{2.13}$$

By formulae (2.5,6b), we have to verify that

$$v^m H^*_m \stackrel{?}{=} (v+sh)^m H^*_m - \Big((v+sh)^m - v^m\Big) H^*_m, \tag{2.14}$$

which is obviously true.

3 The Reduction Map Is Hamiltonian

We need to verify that the map (2.2),

$$A^*_m = \frac{(v+sh)^{m+1} - v^{m+1}}{s(m+1)}, \quad m \in \mathbf{Z}_{\geq 0}, \tag{3.1}$$

is Hamiltonian between the Hamiltonian matrices

$$b = \begin{pmatrix} 0 & \partial \\ \partial & -s\partial \end{pmatrix} \tag{3.2}$$

and

$$B_{nm} = nA_{n+m-1}\partial + \partial mA_{n+m-1}. \tag{3.3}$$

This is equivalent to the equality

$$B^* \stackrel{?}{=} JbJ^t, \tag{3.4}$$

where J is the Fréchet Jacobian of the map (3.1):

$$J_{n,h} = \frac{\partial A^*_n}{\partial h} = A^*_{n,h}, \ J_{n,v} = \frac{\partial A^*_n}{\partial v} = A^*_{n,v}. \tag{3.5}$$

In components, the equality (3.4) becomes:

$$\begin{aligned} &nA^*_{n+m-1}\partial + \partial mA^*_{n+m-1} \stackrel{?}{=} \\ &= A^*_{n,v}\partial A^*_{m,h} + (A^*_{n,h} - sA^*_{n,v})\partial A^*_{m,v}, \quad n,m \in \mathbf{Z}_{\geq 0}. \end{aligned} \tag{3.6}$$

This identity in turn, splits into the pair:

$$(n+m)A^*_{n+m-1} \stackrel{?}{=} A^*_{n,v}A^*_{m,h} + (A^*_{n,h} - sA^*_{n,v})A^*_{m,v}, \tag{3.7}$$

$$mA^*_{n+m-1,x} \stackrel{?}{=} A^*_{n,v}(A^*_{m,h})_x + (A^*_{n,h} - sA^*_{n,v})(A^*_{m,v})_x. \tag{3.8}$$

We start with the identify (3.7). From formulae (2.3,4) we have:

$$A^*_{n,h} - sA^*_{n,v} = v^n. \tag{3.9}$$

Denote

$$\sigma = v + sh. \tag{3.10}$$

Formula (3.7) becomes in view of formula (2.2):

$$(n+m)\frac{\sigma^{n+m} - v^{n+m}}{s(n+m)} \stackrel{?}{=} \frac{\sigma^n - v^n}{s}\sigma^m + v^n\frac{\sigma^m - v^m}{s}, \tag{3.11}$$

which is obviously true.

Next, formula (3.8) becomes:

$$m\left(\frac{\sigma^{n+m} - v^{n+m}}{s(n+m)}\right)_x \stackrel{?}{=} \frac{\sigma^n - v^n}{s}(\sigma^m)_x + v^n\left(\frac{\sigma^m - v^m}{s}\right)_x. \tag{3.12}$$

Since

$$\frac{\partial\sigma}{\partial v} = 1, \tag{3.13}$$

the identity (3.12) splits into the pair:

$$m\sigma^{n+m-1}\sigma_h h_x \stackrel{?}{=} (\sigma^n - v^n)m\sigma^{m-1}\sigma_h h_x + v^n m\sigma^{m-1}\sigma_h h_x, \tag{3.14}$$

$$m(\sigma^{n+m-1} - v^{n+m-1})v_x \stackrel{?}{=} (\sigma^n - v^n)m\sigma^{m-1}v_x + v^n m(\sigma^{m-1} - v^{m-1})v_x, \tag{3.15}$$

and each one of these identities is obviously true.

4 Other Two-Dimensional Systems

Free-surface systems, such as (1.11), are naturally attached to local Lie algebras, in particular to Poisson manifolds [4]. In the two-dimensional case, the general form of systems liftable into the space of moments has the form [4]

$$h_t = Q_m A_m + P_m A_{m,x} \tag{4.1a}$$

$$u_t = P_m u^m u_x + \bar{P}_m u^m - u_y \int_0^y dy(P_m(u^m)_x + Q_m u^m), \tag{4.1b}$$

where $P_m, \bar{P}_m, Q_m$ are arbitrary functions of x and the A_n's; the resulting evolution for the moments is:

$$A_{n,t} = nA_{n+m-1}\bar{P}_m + A_{n+m}Q_m + A_{n+m,x}P_m. \tag{4.2}$$

The systems (1.11) we have looked at in the previous Sections are of the above form, with

$$\bar{P}_m = H_{m,x},\ Q_m = (m+1)H_{m+1,x},\quad P_m = (m+1)H_{m+1}. \tag{4.3}$$

Let us determine when the system (4.1) can be self-consistently constrained onto the submanifold $\{u_y = s\}$. Differentiating formula (4.1b) with respect to y, we get:

$$u_{y,t} = P_m(mu^{m-1}u_y u_x + u^m u_{xy}) + \bar{P}_m m u^{m-1}u_y -$$
$$-u_{yy}\int_0^y dy(P_m(u^m)_x + Q_m u^m) - u_y(P_m m u^{m-1}u_x + Q_m u^m) =$$
$$= P_m u^m u_{xy} - u_{yy}\int_0^y dy(P_m(u^m)_x + Q_m u^m) + \tag{4.4a}$$
$$+ u_y u^m((m+1)\bar{P}_{m+1} - Q_m). \tag{4.4b}$$

Hence, the system (4.1) is constrainable iff

$$Q_m = (m+1)\bar{P}_{m+1},\quad m \in \mathbf{Z}_{\geq 0}. \tag{4.5}$$

The resulting h, v-system can be read off formulae (4.1) for $y = 0$:

$$h_t = (m+1)\bar{P}^*_{m+1}A^*_m + P^*_m A^*_{m,x}, \tag{4.6a}$$
$$v_t = P^*_m v^m v_x + \bar{P}^*_m v^m. \tag{4.6b}$$

Formula (4.3) shows that the relations (4.5) are satisfied for our original system (1.11).

The system (4.1) is of a *general character*. Among *Hamiltonian* systems of this type, there exists a two-parameter family [2, formula (2.99′)] given by the Hamiltonian matrix

$$B^{\alpha,\beta}_{nm} = (\alpha n + \beta)A_{n+m}\partial + \partial(\alpha m + \beta)A_{n+m}, \tag{4.7}$$

where α and β are arbitrary constants. For this case, we have:

$$A_{n,t} = B^{\alpha,\beta}_{nm}(H_m) = (\alpha n + \beta)A_{n+m}H_{m,x} + (\alpha m + \beta)(A_{n+m}H_m)_x =$$
$$= nA_{n+m}\alpha H_{m,x} + A_{n+m}(\alpha m + 2\beta)H_{m,x} + A_{n+m,x}(\alpha m + \beta)H_m. \tag{4.8}$$

From formula (4.2) we see that

$$\bar{P}_0 = 0;\ \bar{P}_{m+1} = \alpha H_{m,x};\ Q_m = (\alpha m + 2\beta)H_{m,x};\ P_m = (\alpha m + \beta)H_m. \tag{4.9}$$

Therefore, the constrainability criterion (4.5) is satisfied provided

$$(\alpha m + 2\beta) H_{m,x} = (m+1)\alpha H_{m,x}, \quad m \in \mathbf{Z}_{\geq 0}, \tag{4.10a}$$

or

$$(\alpha m + 2\beta) = (m+1)\alpha, \quad m \in \mathbf{Z}_{\geq 0}, \tag{4.10b}$$

and this happens iff

$$\alpha = 2\beta. \tag{4.11}$$

This is a very puzzling result. To see why, notice that the Hamiltonian matrix $B^{\alpha,\beta}$ (4.7) is *linear* in the field variables (the A_n's). Hence, it corresponds to a Lie algebra. An easy calculation shows that this Lie algebra has the commutator

$$[\boldsymbol{X}, \boldsymbol{Y}]_k = \sum_{n+m=k} ((\alpha n + \beta) X_n Y_{m,x} - (\alpha m + \beta) Y_m X_{n,x}). \tag{4.12}$$

Setting

$$f(x,p) = \sum_{n \geq 0} X_n p^n, \quad g(x,p) = \sum_{m \geq 0} Y_m p^m, \tag{4.13}$$

we can convert the commutator (4.12) into the following Poisson bracket on $\mathbf{R}^2$:

$$\{f, g\} = \beta(f g_x - f_x g) + \alpha p (f_p g_x - f_x g_p). \tag{4.14}$$

There is *nothing* in this Poisson bracket to indicate that the ratio

$$\alpha : \beta = 2 : 1 \tag{4.15}$$

is distinguished from all the other ratios.

Let us now consider what happens with system (4.8) for the case

$$\alpha = 2, \ \beta = 1, \tag{4.16}$$

when this system is restricted onto the submanifolds $\{u_y = s\}$. By formula (4.9), the full system (4.1) has the form:

$$h_t = A_m H_{m,x} + (2m+1)(A_m H_m)_x, \tag{4.17a}$$

$$u_t = (2m+1) u^m u_x H_m + 2u^{m+1} H_{m,x} - u_y \int_0^y dy((2m+1)(u^m)_x H_m +$$

$$+ 2(m+1) u^m H_{m,x}). \tag{4.17b}$$

Hence, the restricted system becomes:

$$h_t = A^*_m H^*_{m,x} + (2m+1)(A^*_m H^*_m)_x, \tag{4.18a}$$
$$v_t = (2m+1)v^m v_x H^*_m + 2v^{m+1} H^*_{m,x}. \tag{4.18b}$$

Proposition 4.19. (i) The system (4.18) can be put into the following form:

$$\begin{pmatrix} h \\ \\ v \end{pmatrix}_t = \begin{pmatrix} h\partial + \partial h & v\partial + \partial v \\ \\ v\partial + \partial v & -s(v\partial + \partial v) \end{pmatrix} \begin{pmatrix} \delta H^*/\delta h \\ \\ \delta H^*/\delta v \end{pmatrix}; \tag{4.20a}$$

(ii) The matrix

$$b = \begin{pmatrix} h\partial + \partial h & v\partial + \partial v \\ \\ v\partial + \partial v & -s(v\partial + \partial v) \end{pmatrix} \tag{4.20b}$$

is Hamiltonian;
(iii) The reduction map $A^*_n = \int_0^h (v+sy)^n dy$ (2.2) is Hamiltonian between the Hamiltonian matrices b (4.20b) and $B^{2,1}$ (4.6).
Proof. (i) We have to verify that

$$(h\partial + \partial h)(H^*_h) + (v\partial + \partial v)(H^*_v) \overset{?}{=} (A^*_m\partial + (2m+1)\partial A^*_m)(H^*_m), \tag{4.21a}$$
$$(v\partial + \partial v)(H^*_h) - s(v\partial + \partial v)(H^*_v) \overset{?}{=} ((2m+1)v^m v_x + 2v^{m+1}\partial)(H^*_m). \tag{4.21b}$$

Formulae (2.5, 6b, 13,) and (3.10) transform the identities (4.21) into the form:

$$(h\partial + \partial h)\sigma^m + (v\partial + \partial v)\frac{\sigma^m - v^m}{s} \overset{?}{=} A^*_m\partial + (2m+1)\partial A^*_m, \tag{4.22a}$$
$$(v\partial + \partial v)v^m \overset{?}{=} (2m+1)v^m v_x + 2v^{m+1}\partial. \tag{4.22b}$$

Formula (4.22b) is obvious.
Formula (4.22a) can be rewritten as

$$(\sigma\partial+\partial\sigma)\sigma^m - (v\partial+\partial v)v^m \overset{?}{=} \frac{\sigma^{m+1} - v^{m+1}}{m+1}(2m-2)\partial + (2m+1)\left(\frac{\sigma^{m+1} - v^{m+1}}{m+1}\right)_x, \tag{4.23}$$

and it follows from formula (4.22b);

(ii) Let $\mathcal{G}$ be a Lie algebra. It acts by derivations on itself. Hence, we can form the semidirect sum Lie algebra, with the commutator

$$\left(\begin{array}{c} x_1 \\ \\ y_1 \end{array}\right), \left(\begin{array}{c} x_2 \\ \\ y_2 \end{array}\right) = \left(\begin{array}{c} [x_1, x_2] \\ \\ [x_1, y_2] - [x_2, y_1] + \epsilon[y_1, y_2] \end{array}\right) \tag{4.24}$$

where ϵ is an arbitrary constant that, when nonzero, can be scaled away to $\epsilon = 1$. The Hamiltonian matrix corresponding to the commutator (4.24) has the form

$$\left(\begin{array}{cc} B_h & B_v \\ \\ B_v & \epsilon B_v \end{array}\right), \tag{4.25}$$

where $\epsilon = -s$ and $B = B_q = B_q(\mathcal{G})$ is the Hamiltonian matrix naturally attached to the Lie algebra $\mathcal{G}$ with the dual coordinates on $\mathcal{G}^*$ denoted by q. Our matrix b (4.20b) is of the form (4.25), with $\mathcal{G} = \mathcal{D}(\mathbf{R}^1)$ being the Lie algebra of vector fields on $\mathbf{R}^1$;

(iii) We have to verify the identity

$$JbJ^t \stackrel{?}{=} B^{2,1*}, \tag{4.26}$$

where the matrix b is given by formula (4.20b). In components, this identity becomes:

$$\begin{aligned} &(A^*_{n,h}(h\partial + \partial h) + A^*_{n,v}(v\partial + \partial v))A^*_{m,h} + v^n(v\partial + \partial v)A^*_{m,v} \stackrel{?}{=} \\ &\stackrel{?}{=} (2n+1)A^*_{n+m}\partial + \partial(2m+1)A^*_{n+m}. \end{aligned} \tag{4.27}$$

By formulae (2.2-4), this can be rewritten as

$$\begin{aligned} &(\sigma^n(h\partial + \partial h) + \frac{\sigma^n - v^n}{s}(v\partial + \partial v)\sigma^m + v^n(v\partial + \partial v)\frac{\sigma^m - v^m}{s} \stackrel{?}{=} \\ &\stackrel{?}{=} (2n+1)\frac{\sigma^{n+m+1} - v^{n+m+1}}{(n+m+1)s}\partial + \partial(2m+1)\frac{\sigma^{n+m+1} - v^{n+m+1}}{(n+m+1)s} = \\ &= \frac{2}{s}(\sigma^{n+m+1} - v^{n+m+1})\partial + \frac{2m+1}{s}(\sigma^{n+m}\sigma_x - v^{n+m}v_x), \end{aligned} \tag{4.28}$$

or as

$$\begin{aligned} &(\sigma^n(\sigma\partial + \partial\sigma) - v^n(v\partial + \partial v))\sigma^m + v^n(v\partial + \partial v)(\sigma^m - v^m) = \\ &= \sigma^n(\sigma\partial + \partial\sigma)\sigma^m - v^n(v\partial + \partial v)v^m \stackrel{?}{=} \\ &\stackrel{?}{=} 2(\sigma^{n+m+1} - v^{n+m+1})\partial + (2m+1)(\sigma^{n+m}\sigma_x - v^{n+m}v_x, \end{aligned} \tag{4.29}$$

which follows from the single equality

$$w^n(w\partial + \partial w)w^m = 2w^{n+m+1}\partial + (2m+1)w^{n+m}w_x, \tag{4.30}$$

which in turn follows from formula (4.22b) or is obvious in its own right ■

Remark 4.31. Formulae in [4,5] suggest - but not prove - that the constant-shear reductions of Hamiltonian systems do not exist in the $N+1$ dimension for $N \neq 1$.

5 Lie Algebra Of Reducible Flows

A reducible dynamical system (4.2) in the momentum space has, by formula (4.5), the form

$$\hat{X}(A_n) = A_{n,t} = (n+m)A_{n+m-1}\bar{P}_m + A_{n+m,x}P_m. \tag{5.1}$$

Suppose we have another reducible vector field, $\hat{Y}$:

$$\hat{Y}(A_n) = (n+m)A_{n+m-1}\bar{\Phi}_m + A_{n+m,x}\Phi_m. \tag{5.2}$$

Theorem 5.3. The commutator of reducible vector fields is again reducible.

Proof. We shall show that

$$[\hat{X}, \hat{Y}](A_n) = (n+r)A_{n+r-1}\bar{\Omega}_r + A_{n+r,x}\Omega_r, \tag{5.4}$$

where

$$\bar{\Omega}_r = \hat{X}(\bar{\Phi}_r) - \hat{Y}(\bar{P}_r) + \sum_{k+m=r+1} \bar{\Phi}_k \bar{P}_m +$$

$$+ \sum_{k+m=r} (\Phi_k \bar{P}_{m,x} - P_k \bar{\Phi}_{m,x}), \tag{5.5a}$$

$$\Omega_r = \hat{X}(\Phi_r) - \hat{Y}(P_r) + \sum_{k+m=r+1} (k\Phi_k \bar{P}_m - mP_m\bar{\Phi}_k) +$$

$$+ \sum_{k+m=r} (\Phi_k P_{m,x} - P_m \Phi_{k,x}). \tag{5.5b}$$

We have:

$$[\hat{X}, \hat{Y}](A_n) - (n+r)A_{n+r-1}[\hat{X}(\bar{\Phi}_r) - \hat{Y}(\bar{P}_r)] - \tag{5.6a}$$

$$-A_{n+r,x}[\hat{X}(\Phi_r) - \hat{Y}(P_r)] = \tag{5.6b}$$

$$= \bar{\Phi}_k(n+k)\hat{X}(A_{n+k-1}) + \Phi_k[\hat{X}(A_{n+k})]_x -$$

$$-\bar{P}_m(n+m)\hat{Y}(A_{n+m-1}) - P_m[\hat{Y}(A_{n+m})]_x =$$

$$= \bar{\Phi}_k(n+k)[(n+k-1+m)A_{n+k-1+m-1})\bar{P}_m + A_{n+k-1+m,x}P_m] + \tag{5.7a}$$

$$+\Phi_k[(n+k+m)(A_{n+k+m-1,x}\bar{P}_m + A_{n+k+m-1}\bar{P}_{m,x}) + \tag{5.7b}$$

$$+A_{n+k+m,xx}P_m + A_{n+k+m,x}P_{m,x}] - \tag{5.7c}$$

$$-\bar{P}_m(n+m)[(n+m-1+k)A_{n+m-1+k-1}\bar{\Phi}_k + A_{n+m-1+k,x}\bar{\Phi}_k] - \tag{5.8a}$$

$$-P_m[(n+m+k)(A_{n+m+k-1,x}\bar{\Phi}_k + A_{n+m+k-1}\bar{\Phi}_{k,x}) + \tag{5.8b}$$

$$+A_{n+m+k,xx}\Phi_k + A_{n+m+k,x}\Phi_{k,x}]. \tag{5.8c}$$

The first summands in (5.7c) and (5.8c) cancel out.

The second summands in (5.7a) and (5.8a), and the first summands in (5.7b) and (5.8b), combine into

$$A_{n+r,x} \sum_{k+m=r+1} (k\Phi_k \bar{P}_m - mP_m \bar{\Phi}_k), \tag{5.9}$$

while the second summands in (5.7c) and (5.8c) yield

$$A_{n+r,x} \sum_{k+m=r} (\Phi_k P_{m,x} - P_m \Phi_{k,x}). \tag{5.10}$$

Formulae (5.6b, 9, 10) account for formula (5.5b).

What remains, the first summands in (5.7a) and (5.8a), and the second summands in (5.7b) and (5.8b), combine into

$$\begin{aligned}&(n+r)A_{n+r-1}\{ \sum_{k+m=r+1} [(n+k)\bar{\Phi}_k \bar{P}_m - (n+m)\bar{P}_m \bar{\Phi}_k] + \\ &+ \sum_{k+m=r} (\Phi_k \bar{P}_{m,x} - P_m \bar{\Phi}_{k,x})\}\end{aligned} \tag{5.11}$$

which, together with the second summand in (5.6a), account for formula (5.5a) ■

Since the momentum map

$$A_n = \int_0^h u^n dy, \quad n \in \mathbf{Z}_{\geq 0}, \tag{5.12}$$

is injective, Theorem 5.3 implies that the 2+1-dimensional hydrodynamic systems (4.1) that are reducible in the physical space,

$$h_t = (m+1)\bar{P}_{m+1} A_m + P_m A_{m,x}, \tag{5.13a}$$

$$u_t = P_m u^m u_x + \bar{P}_m u^m - u^y \int_0^y dy (P_m (u^m)_x + (m+1)\bar{P}_{m+1} u^m), \tag{5.13b}$$

also form a Lie algebra. In particular, when there is no x-dependence, so that the P_m's vanish and the $\bar{P}_m$'s depend on the A_n's but not on the derivatives of the A_n's, we get a free-surface analog of the Lie algebra ODE's:

$$h_t = (m+1)\bar{P}_{m+1} A_m, \tag{5.14a}$$

$$u_t = \bar{P}_0 + \bar{P}_{m+1}(u^{m+1} - u_y \int_0^y dy (m+1) u^m). \tag{5.14b}$$

All these Lie algebras possess subalgebras induced by the constain

$$\{h = A_0 = A_1 = ... = 0\}. \tag{5.15}$$

Remark 5.16. The 2+1-dimensional free-surface hydrodynamics is infused with Lie algebras, such as (4.1), (4.2), (5.1), (5.13). This picture is very different from the theory of systems of hydrodynamical type in $1+1-d$, of the form

$$u_{i,t} = \sum_j \mathcal{M}_i^j (\boldsymbol{u}) u_{j,x}, \tag{5.17}$$

where, in general, a commutator of two such systems is no longer of hydrodynamic type (see [8]).

6 Multilayer Representations

Imagine that the Benney system (1.5)

$$h_t = (\int_0^h u dy)_x, \tag{6.1a}$$

$$u_t = uu_x + gh_x - u_y \int_0^y u_x dy, \tag{6.1b}$$

is broken into N layers

$$h_{k-1} \leq y \leq h_k, \quad k = 1, ..., N, \ h_0 = 0, \tag{6.2}$$

such that in each layer the velocity profile u_k is y-independent. The Benney system them turns into the $2N$-component system

$$h_{k,t} = (u_k h_k)_x, \tag{6.3a}$$

$$u_{k,t} = u_k u_{k,x} + gh_x, \quad k = 1, ..., N, \tag{6.3b}$$

$$h = \sum_{k=1}^{n} h_k. \tag{6.3c}$$

This idea and the system (6.3) are due to Zakharov [10], who in addition showed that, rather mysteriously, the system (6.3) appears also as the zero-dispersion limit of a vector Nonlinear Schrödinger equation.

The system (6.3) can be considered as a multi-component version of the classical long-wave system (1.1), and it was analyzed in detail by Pavlov and Tsarev [9]. As Zakharov noted, the moment map (1.6)

$$A_n == \int_0^h u^n dy, \quad n \in \mathbf{Z}_{\geq 0}, \tag{6.4}$$

which now becomes

$$A_n = \sum_{k=1}^{N} h_k u_k^n, \quad n \in \mathbf{Z}_{\geq 0}, \tag{6.5}$$

maps the $2N$-component system (6.3) into the infinite-component Benney momentum system (1.7)

$$A_{n,t} = A_{n+1,x} + gnA_{n-1}A_{0,x}, \quad n \in \mathbf{Z}_{\geq 0}. \tag{6.6}$$

It's easy to see that the same conclusion applies to all the higher Benney flows constructed in [6, 7], and indeed to *any* flow (1.9) in the Hamiltonian structures (1.10), (4.7) or any other *linear* Hamiltonian structure. The argument is as follows.

Let $\mathcal{G}$ be a Lie algebra, C_A the (differential-difference) ring of functions on $\mathcal{G}^*$, and $B = B_A$ the natural Hamiltonian structure in C_A attached to $\mathcal{G}$ (see [5].) Let $\mathcal{G}^{<N>}$ be the direct sum of N copies of $\mathcal{G}$. The homomorphism of Lie algebras

$$\varphi : \mathcal{G}^{<N>} \ \rightarrow \ \mathcal{G}, \tag{6.7a}$$

$$\varphi\Big(\oplus_{k=1}^{N} x_k \Big) = \sum_{k=1}^{N} x_k, \tag{6.7b}$$

induces the corresponding linear Hamiltonian map

$$\Phi : C_A \ \rightarrow \ C_A^{<N>},$$

$$\Phi(A_n) = \sum_{k=1}^{N} A_{n|k}. \tag{6.8}$$

If C_V is another ring, with a Hamiltonian structure on it, and if

$$w : C_A \ \rightarrow \ C_V \tag{6.9}$$

is a Hamiltonian map, then so is its k^{th}-copy version:

$$w_k : C_{A_{|k}} \ \rightarrow \ C_{V_{|k}}. \tag{6.10}$$

The composition

$$Z = \Phi \circ \Omega : \ C_A \ \rightarrow \ C_V^{<N>}, \tag{6.11}$$

$$\Omega = \oplus_{k=1}^{N} w_k, \ C_V^{<N>} = \oplus_{k=1}^{N} C_{V_{|k}}, \tag{6.12}$$

is then a multilayer analog of the single-layer canonical map (6.9).

In particular, for the Hamiltonian matrix (1.10)

$$h_t = \partial(m A_{m-1} H_m) \tag{6.13a}$$

$$u_t = \partial(u^m H_m) - u_y \int_0^y dy (m u^{m-1} H_m)_x. \tag{6.13b}$$

with the Hamiltonian map (1.13)

$$w(A_n) = h u^n, \tag{6.14}$$

the general construction above gives:

$$h_{k,t} = \partial \frac{\delta}{\delta u_k} Z(H), \tag{6.15a}$$

$$u_{k,t} = \partial \frac{\delta}{\delta h_k} Z(H), \tag{6.15b}$$

$$Z(A_n) = \sum_{k=1}^{N} h_k u_k^n. \tag{6.15c}$$

If H is the 2^{nd} Hamiltonian (1.8),

$$H = \frac{1}{2}(A_2 + gA_0^2) \tag{6.16}$$

then

$$Z(H) = \frac{1}{2}(\sum_k h_k u_k^2 + g(\sum_k h_k)^2), \tag{6.17a}$$

$$\frac{\delta Z(H)}{\delta u_k} = h_k u_k, \quad \frac{\delta Z(H)}{\delta h_k} = \frac{u_k^2}{2} + gh, \tag{6.17b}$$

and we recover the Zakharov system (6.3). If H is the 3^{rd} Hamiltonian,

$$H = \frac{1}{3}A_3 + gA_0A_1, \tag{6.18}$$

then

$$Z(H) = \frac{1}{3}\sum_k h_k u_k^3 + g(\sum_k h_k u_k)(\sum_\ell h_\ell), \tag{6.19a}$$

$$\frac{\delta Z(H)}{\delta u_k} = h_k u_k^2, \ \frac{\delta Z(H)}{\delta h_k} = \frac{u_k^3}{3} + gu_k h + gA_1 \Rightarrow \tag{6.19b}$$

$$h_{k,t} = (h_k u_k^2)_x, \tag{6.20a}$$

$$u_{k,t} = u_k^2 u_{k,x} + g(u_k h + A_1)_x, \quad k = 1, ..., N. \tag{6.20b}$$

All the odd-numbered higher Benney flows can be restricted onto the invariant submanifold [3]

$$\{A_1 = A_3 = A_5 = ... = 0\}. \tag{6.21}$$

The analog of this fact is this: Suppose N is even:

$$N = 2M, \tag{6.22a}$$

and

$$h_{k+m} = h_k, \ u_{k+m} = -u_k, \quad k = 1, ..., M. \tag{6.22b}$$

Then all the odd flows in the (h, u)-space can be properly reduced by the constrain (6.22). The 3^{rd} flow (6.20) becomes:

$$h_{k,t} = (h_k u_k^2)_x, \tag{6.23a}$$

$$u_{k,t} = u_k^2 u_x + 2g(u_x h)_x, \quad k = 1, ..., M, \tag{6.23b}$$

in the variables

$$h_k = h_k, \ U_k = u_k^2, \quad k = 1, ..., M, \tag{6.24}$$

$$h_{k,t} = (h_k U_k)_x, \tag{6.25a}$$
$$U_{k,t} = U_k U_{k,x} + 2g(U_{k,x} h + 2U_k h_x), \quad k = 1, ..., M. \tag{6.25b}$$

Similar Hamiltonian construction applies to the case of constant shear. Equations (6.15) become

$$h_{k,t} = \partial \frac{\delta}{\delta v_k} Z(H), \tag{6.26a}$$
$$v_{k,t} = \partial \left(\frac{\delta}{\delta h_k} - s_k \frac{\delta}{\delta v_k} \right) Z(H), \quad k = 1, ..., N, \tag{6.26b}$$

where now

$$Z(A_n) = \sum_k \int_0^{h_k} (v_k + s_k y)^n dy = \sum_k \frac{(v_k + s_k h_k)^{n+1} - v_k^{n+1}}{s_k(n+1)}. \tag{6.27}$$

In particular, for the 2^{nd} flow, we get:

$$h_{k,t} = (h_k v_k + \frac{s_k}{2} h_k^2)_x, \tag{6.28a}$$
$$v_{k,t} = v_k v_{k,x} + g h_x, \quad k = 1, ..., N. \tag{6.28b}$$

This is a multicomponent version of the single component shear system (1.18), and a shear extension of the Zakharov system (6.3).

The shear analog of the constrain (6.22) now becomes:

$$h_{k+m} = h_k, \quad v_{k+m} = -v_k, \quad s_{k+M} = -s_k, \quad k = 1, ..., M. \tag{6.29}$$

The construction above works only for systems with *linear* Hamiltonian structures. It doesn't apply to the hydrodynamic chain [3]

$$A_{n,t} = A_{n+1,x} + (an + b) A_n A_{0,x} + c A_0 A_{n,x}, \ n \in \mathbf{Z}_{\geq 0}, \tag{6.30a}$$
$$a, b, c = const, \tag{6.30b}$$

with

$$b \neq 2c. \tag{6.30c}$$

Nevertheless, we now show that the Zakharov map (6.5) and its shear version (6.27) work in the most general circumstances.

Theorem 6.31. (i) The Zakharov map (6.5) applies to *any* flow (4.2):

$$A_{n,t} = n A_{n+m-1} \bar{P}_m + A_{n+m} Q_m + A_{n+m,x} P_m. \tag{6.32}$$

(ii) The map (6.27) applies to *any* flow (5.1):

$$A_{n,t} = (n+m) A_{n+m-1} \bar{P}_m + A_{n+m,x} P_m. \tag{6.33}$$

Proof. The idea is this. Either of the two maps is a local diffeomorphism between the variables $h_1, u_1, ..., h_N, u_N$ and $A_0, ..., A_{2N-1}$. Therefore, we can consider these maps as providing an anvariant submanifold for each flow. This means that if we find $-$ by whatever means $-$ an evolution in the $\{u_k, h_k\}-$space that is properly embedded into the evolution in the $A-$space, that's it.

Now for the details. Writing $(\cdot)^{\cdot}$ instead of $(\cdot)_t$, we convert, first for the Zakharov case, formula (4.2) into

$$\sum_k (\dot{h}_k u_k^n + n h_k u_k^{n-1} \dot{u}_k) =$$
$$= \sum_k \{ n h_k u_k^{n+m-1} \bar{P}_m^* + h_k u_k^{n+m} Q_m^* + [h_{k,x} u_k^{n+m} + (n+m) h_k u_k^{n+m-1} u_{k,x}] P_m^* \}, \tag{6.34}$$

where

$$(\cdot)^* = Z(\cdot), \;\; Z(A_n) = \sum_k h_k u_k^n, \quad n \in \mathbf{Z}_{\geq 0}. \tag{6.35}$$

We drop the $\sum_k$ operator from each side, multiply the resulting equation by u_k^{-n}, and find:

$$\dot{h}_k + n h_k u_k^{-1} \dot{u}_k =$$
$$= h_k u_k^m Q_m^* + [h_{k,x} u_k^m + m h_k u_k^{m-1} u_{k,x}] P_m^* + \tag{6.36a}$$
$$+ n h_k u_k^{-1} [u_k^m \bar{P}_m^* + u_k^m u_{k,x} P_m^*], \tag{6.36b}$$

whence

$$\dot{h}_k = h_k u_k^m Q_m^* + [h_{k,x} u_k^m + m h_k u_k^{m-1} u_{k,x}] P_m^*, \tag{6.37a}$$
$$\dot{u}_k = u_k^m \bar{P}_m^* + u_k^m u_{k,x} P_m^*. \tag{6.37b}$$

Next, denoting

$$\sigma_k = v_k + s_k h_k, \tag{6.38}$$

so that

$$Z(A_n) = \sum_k \frac{\sigma_k^{n+1} - v_k^{n+1}}{(n+1) s_k}, \tag{6.39}$$

we convert formula (5.1) into

$$\dot{A}_n = \sum_k \frac{(\sigma_k^{n+1} - v_k^{n+1})^{\cdot}}{(n+1) s_k} = \sum_k \{ \sigma_k^n \dot{h}_k + \frac{\sigma_k^n - v_k^n}{s_k} \dot{v}_k \} =$$
$$= \sum_k \{ \frac{\sigma_k^{n+m} - v_k^{n+m}}{s_k} \bar{P}_m^* + \left(\frac{\sigma_k^{n+m+1} - v_k^{n+m+1}}{(n+m+1) s_k} \right)_x P_m^* \}, \tag{6.40}$$

where again

$$(\cdot)^* = Z(\cdot), \tag{6.41}$$

but with the map Z given by formula (6.39).

We now drop the operator $\sum_k$ from each side of formula (6.40) and get, for each k, the equation

$$\sigma_k^n \dot{h}_k + \frac{\sigma_k^n - v_k^n}{s_k}\dot{v}_k = = \frac{\sigma_k^{n+m} - v_k^{n+m}}{s_k}\bar{P}_m^* + \left(\frac{\sigma_k^{n+m+1} - v_k^{n+m+1}}{(n+m+1)s_k}\right)_x P_m^*, \tag{6.42}$$

For $n = 0$, equation (6.42) yields:

$$\dot{h}_k = \frac{\sigma_k^m - v_k^m}{s_k}\bar{P}_m^* + \left(\frac{\sigma_k^{m+1} - v_k^{m+1}}{(m+1)s_k}\right)_x P_m^*. \tag{6.43}$$

Substituting this back into equation (6.42), we find:

$$\frac{\sigma_k^n - v_k^n}{s_k}\dot{v}_k = [\sigma_k^{n+m} - v_k^{n+m} - \sigma_k^n(\sigma_k^m - v_k^m)]s_k^{-1}\bar{P}_m^* +$$

$$+[(\sigma_k^{n+m} - \sigma_k^n\sigma_k^m)\sigma_{k,x} - (v_k^{n+m} - \sigma_k^n v_k^m)v_{k,x}]s_k^{-1}P_m^* \Leftrightarrow \tag{6.44}$$

$$\dot{v}_k = v_k^m \bar{P}_m^* + v_k^m v_{k,x} P_m^* \quad \blacksquare \tag{6.45}$$

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